

# ON NONLINEAR SCHRÖDINGER EQUATIONS WITH ALMOST PERIODIC INITIAL DATA

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**ABSTRACT.** We consider the Cauchy problem of nonlinear Schrödinger equations (NLS) with almost periodic functions as initial data. We first prove that, given a frequency set  $\omega = \{\omega_j\}_{j=1}^\infty$ , NLS is local well-posed in the algebra  $\mathcal{A}_\omega(\mathbb{R})$  of almost periodic functions with absolutely convergent Fourier series. Then, we prove a finite time blowup result for NLS with a nonlinearity  $|u|^p$ ,  $p \in 2\mathbb{N}$ . This provides the first instance of finite time blowup solutions to NLS with generic almost periodic initial data.

## 1. INTRODUCTION

We consider the Cauchy problem of the following nonlinear Schrödinger equation (NLS) with an algebraic power-type nonlinearity:

$$\begin{cases} i\partial_t u + \partial_x^2 u = \mathcal{N}(u), \\ u|_{t=0} = f, \end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \quad (1.1)$$

where the nonlinearity is given by  $\mathcal{N}(u) = \mathcal{N}_p(u, \bar{u}) = u^k \bar{u}^{p-k}$ ,  $0 \leq k \leq p$ ,  $p \in \mathbb{N}$ ,  $k \in \mathbb{N} \cup \{0\}$ . For example, this includes the standard power nonlinearity  $|u|^{p-1}u$  and a nonlinearity  $|u|^p$  without gauge invariance.

The Cauchy problem (1.1) has been studied extensively in terms of the usual Sobolev spaces  $H^s(\mathbb{R})$  on the real line and the Sobolev spaces  $H_{\text{per}}^s(\mathbb{R}) \simeq H^s(\mathbb{T})$  of periodic functions (of a fixed period) on  $\mathbb{R}$ . See [4, 13] for the references therein. Our main interest in this paper is to study the Cauchy problem (1.1) with *almost periodic* functions as initial data.

**Definition 1.1.** We say that a complex-valued function  $f$  on  $\mathbb{R}$  is *almost periodic*, if it is continuous and, for every  $\varepsilon > 0$ , there exists  $L = L(\varepsilon, f) > 0$  such that every interval of length  $L$  on  $\mathbb{R}$  contains a number  $\tau$  such that

$$\sup_{x \in \mathbb{R}} |f(x - \tau) - f(x)| < \varepsilon.$$

We use  $AP(\mathbb{R})$  to denote the space of almost periodic functions on  $\mathbb{R}$ .

The study of almost periodic functions was initiated by Bohr [2]. In the following, we briefly go over the basic properties of almost periodic functions. See Besicovitch [1], Corduneanu [5], and Katznelson [9] for more on the subject. Let us first state several equivalent characterizations for almost periodic functions.

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**Definition 1.2.** (i) We say that a function  $f$  on  $\mathbb{R}$  has the *approximation property*, if it can be uniformly approximated by trigonometric polynomials. More precisely, given any  $\varepsilon > 0$ , there exists a trigonometric polynomial  $P_\varepsilon(x)$  such that

$$\sup_{x \in \mathbb{R}} |f(x) - P_\varepsilon(x)| < \varepsilon.$$

(ii) We say that a continuous function on  $\mathbb{R}$  is *normal* if, given any  $\{x_n\}_{n=1}^\infty \subset \mathbb{R}$ , the collection  $\{f(\cdot + x_n)\}_{n=1}^\infty$  is precompact in  $L^\infty(\mathbb{R})$ . Namely, there exists a subsequence  $\{f(\cdot + x_{n_j})\}_{j=1}^\infty$  uniformly convergent on  $\mathbb{R}$ .

An important fact is that the set of almost periodic functions, the set of functions with the approximation property, and the set of normal functions all coincide. Hence, we freely use any of these three characterizations in the following. We also point out that these three notions can be extended to Banach-space valued functions and that they are also equivalent in the Banach space setting. Given a Banach space  $X$ , we use  $AP(\mathbb{R}; X)$  to denote the space of almost periodic functions on  $\mathbb{R}$  with values in  $X$ .

It is known that  $AP(\mathbb{R})$  is a closed subalgebra of  $L^\infty(\mathbb{R})$  and that almost periodic functions are uniformly continuous. Given  $f \in AP(\mathbb{R})$ , we can define the so-called *mean value*  $M(f)$  of  $f$  by

$$M(f) := \lim_{L \rightarrow \infty} \frac{1}{2L} \int_{-L}^L f(x) dx, \quad (1.2)$$

where the limit on the right-hand side of (1.2) always exists if  $f \in AP(\mathbb{R})$ . Given  $f \in AP(\mathbb{R})$ , we define the  $\mathcal{L}^2$ -norm by the mean value of  $|f|^2$ :

$$\|f\|_{\mathcal{L}^2} := \lim_{L \rightarrow \infty} \left( \frac{1}{2L} \int_{-L}^L |f(x)|^2 dx \right)^{\frac{1}{2}}. \quad (1.3)$$

Note that the limit on the right-hand side of (1.3) exists since the algebra property of  $AP(\mathbb{R})$  states that  $|f|^2$  is almost periodic, if  $f \in AP(\mathbb{R})$ . We have the following lemma.

**Lemma 1.3** (Lemma on p. 177 in [9]). *Let  $f \in AP(\mathbb{R})$  such that  $f \geq 0$  on  $\mathbb{R}$ . If  $f$  is not identically equal to 0, then  $M(f) > 0$ . In particular, the  $\mathcal{L}^2$ -norm, defined in (1.3), of a function  $f \in AP(\mathbb{R})$  is 0 if and only if  $f \equiv 0$ . Hence, it is indeed a norm on  $AP(\mathbb{R})$ .*

Note that the claim of Lemma 1.3 does not hold in general, if  $f \notin AP(\mathbb{R})$ . For example, we have  $M(f) = 0$  for any bounded function  $f$  with a compact support.

Next, we define an inner product  $\langle \cdot, \cdot \rangle_{\mathcal{L}^2}$  on  $AP(\mathbb{R})$  by

$$\langle f, g \rangle_{\mathcal{L}^2} := M(f\bar{g}) = \lim_{L \rightarrow \infty} \frac{1}{2L} \int_{-L}^L f(x)\bar{g}(x) dx. \quad (1.4)$$

for  $f, g \in AP(\mathbb{R})$ . This inner product is well defined for  $f, g \in AP(\mathbb{R})$ , since  $f\bar{g}$  is also in  $AP(\mathbb{R})$ . Moreover, it induces the  $\mathcal{L}^2$ -norm defined in (1.3). Therefore, under the inner product  $\langle \cdot, \cdot \rangle_{\mathcal{L}^2}$ , the space  $AP(\mathbb{R})$  of almost periodic functions becomes a pre-Hilbert space (missing completeness).<sup>1</sup>

<sup>1</sup>In this paper, we only consider almost periodic function in Bohr's sense. There are, however, notions of different classes of generalized almost periodic functions due to Stepanov, Weyl, and Besicovitch. The corresponding spaces are denoted by  $S^p, W^p$  and  $B^p$ , respectively. Then, we have  $AP(\mathbb{R}) \subset S^p \subset W^p \subset B^p$ ,  $p \geq 1$ . Moreover, it is known that  $B^2$  is complete with respect to the  $\mathcal{L}^2$ -norm defined in (1.3).

In this pre-Hilbert space, the complex exponentials  $\{e^{i\omega x}\}_{\omega \in \mathbb{R}}$  form an orthonormal family. We now define the Fourier coefficient of  $f \in AP(\mathbb{R})$  by

$$\widehat{f}(\omega) = \langle f, e^{i\omega x} \rangle_{\mathcal{L}^2} = M(fe^{-i\omega x}). \quad (1.5)$$

By Bessel's inequality, we have

$$\sum_{\omega \in \mathbb{R}} |\widehat{f}(\omega)|^2 \leq \|f\|_{\mathcal{L}^2}^2 < \infty.$$

In particular, this implies that  $\widehat{f}(\omega) = 0$  except for countable many values of  $\omega$ 's. Given  $f \in AP(\mathbb{R})$ , we define its frequency set  $\sigma(f)$  by  $\sigma(f) := \{\omega \in \mathbb{R} : \widehat{f}(\omega) \neq 0\}$  and write

$$f(x) \sim \sum_{\omega \in \sigma(f)} \widehat{f}(\omega) e^{i\omega x}, \quad (1.6)$$

where the right-hand side is the Fourier series associated to  $f \in AP(\mathbb{R})$ . It is known that the orthonormal family  $\{e^{i\omega x}\}_{\omega \in \mathbb{R}}$  is complete in the sense that two distinct almost periodic functions have distinct Fourier series. Moreover, we have the Parseval's identity:

$$\|f\|_{\mathcal{L}^2} = \left( \sum_{\omega \in \mathbb{R}} |\widehat{f}(\omega)|^2 \right)^{\frac{1}{2}} \quad (1.7)$$

for  $f \in AP(\mathbb{R})$ . Regarding the actual convergence of the Fourier series to an almost periodic function, we have the following lemma.

**Lemma 1.4** (Theorem 1.20 in [5]). *Let  $f \in AP(\mathbb{R})$ . If the Fourier series associated to  $f$  converges uniformly, then it converges to  $f$ . Namely, we have*

$$f(x) = \sum_{\omega \in \sigma(f)} \widehat{f}(\omega) e^{i\omega x}. \quad (1.8)$$

Given  $\boldsymbol{\omega} = \{\omega_j\}_{j=1}^\infty \in \mathbb{R}^\mathbb{N}$ , we say that  $\boldsymbol{\omega}$  is linear independent if any relation of the form:

$$\sum_{j=1}^N r_j \omega_j = 0, \quad r_j \in \mathbb{Q},$$

implies that  $r_j = 0$ ,  $j = 1, \dots, N$ . Associated to this notion of linear independence, there is an important criterion on the convergence of the Fourier series to a given almost periodic function.

**Lemma 1.5** (Theorem 1.25 in [5]). *Let  $\boldsymbol{\omega} = \{\omega_j\}_{j=1}^\infty \in \mathbb{R}^\mathbb{N}$  be linearly independent. Suppose that  $f \in AP(\mathbb{R})$  satisfies  $\sigma(f) \subset \boldsymbol{\omega}$ . Then, the Fourier series associated to  $f$  converges uniformly. In particular, (1.8) holds.*

Given a set  $S$  of real numbers, we say that a linearly independent set  $\boldsymbol{\omega} = \{\omega_j\}_{j=1}^\infty$  is a basis for the set  $S$ , if every element in  $S$  can be represented as a finite linear combination of elements in  $\boldsymbol{\omega}$  with rational coefficients. Given  $f \in AP(\mathbb{R})$ , we say that a linearly independent set  $\boldsymbol{\omega} = \{\omega_j\}_{j=1}^N$ , allowing the case  $N = \infty$ , is a basis of  $f$ , if it is a basis of the frequency set  $\sigma(f)$  of  $f$ . Lemma 1.14 in [5] guarantees existence of a basis of  $f \in AP(\mathbb{R})$ . We say that a basis  $\boldsymbol{\omega} = \{\omega_j\}_{j=1}^N$  of  $f$  is an integral basis if any element in the frequency set

$\sigma(f)$  can be written as a finite linear combination of elements in  $\omega$  with integer coefficients.<sup>2</sup> If there exists a finite integral basis of  $f$ , i.e.  $N < \infty$ , then we say that the function  $f$  is *quasi-periodic*. In this paper, we consider generic almost periodic functions, i.e.  $N = \infty$ , but the corresponding results also hold for quasi-periodic functions, i.e.  $N < \infty$ .

Fix  $\omega = \{\omega_j\}_{j=1}^\infty \in \mathbb{R}^\mathbb{N}$ . We consider functions  $f \in AP(\mathbb{R})$  with  $\sigma(f) \subset \omega \cdot \mathbb{Z}^\mathbb{N}$  of the form:

$$f(x) \sim \sum_{\mathbf{n} \in \mathbb{Z}^\mathbb{N}} \widehat{f}(\omega \cdot \mathbf{n}) e^{i(\omega \cdot \mathbf{n})x}, \quad (1.9)$$

where  $\mathbf{n} = \{n_j\}_{j=1}^\infty \in \mathbb{Z}^\mathbb{N}$ . We define the algebra  $\mathcal{A}_\omega(\mathbb{R})$  by

$$\mathcal{A}_\omega(\mathbb{R}) = \{f \in AP(\mathbb{R}) : f \text{ is of the form (1.9) and } \|f\|_{\mathcal{A}_\omega(\mathbb{R})} < \infty\},$$

where the  $\mathcal{A}_\omega(\mathbb{R})$ -norm is given by

$$\|f\|_{\mathcal{A}_\omega(\mathbb{R})} = \|\widehat{f}(\omega \cdot \mathbf{n})\|_{\ell^1_\mathbf{n}(\mathbb{Z}^\mathbb{N})}.$$

See Lemma 2.1 below for some properties of  $\mathcal{A}_\omega(\mathbb{R})$ .

**Remark 1.6.** Note that, if  $\omega$  is linearly independent, then it is an integral basis of  $f$ . If  $\omega$  is not linearly independent, then, we may have  $\omega \cdot \mathbf{n}_1 = \omega \cdot \mathbf{n}_2$  for some  $\mathbf{n}_1 \neq \mathbf{n}_2$ . Namely,  $\widehat{f}(\omega \cdot \mathbf{n})$  in (1.9) may not represent a Fourier coefficient of  $f$  defined in (1.5) and (1.6). In this case, the Fourier coefficient  $\widehat{f}(\alpha)$ ,  $\alpha \in \mathbb{R}$ , is given by  $\widehat{f}(\alpha) = \sum_{\omega \cdot \mathbf{n} = \alpha} \widehat{f}(\omega \cdot \mathbf{n})$ . In the following (for example, see Lemma 1.4 below), we proceed, assuming that  $\omega$  is linearly independent. We point out that the results also hold even when  $\omega$  is not linearly independent. It suffices to note that the definition of  $\mathcal{A}_\omega(\mathbb{R})$  guarantees that the Fourier coefficients  $\widehat{f}(\alpha)$  of  $f \in \mathcal{A}_\omega(\mathbb{R})$  is absolutely summable.

We are now ready to state our first result.

**Theorem 1.7.** *Let  $p \in \mathbb{N}$ . Fix  $\omega = \{\omega_j\}_{j=1}^\infty \in \mathbb{R}^\mathbb{N}$ . Then, NLS (1.1) is locally well-posed in  $\mathcal{A}_\omega(\mathbb{R})$ . More precisely, given  $f \in \mathcal{A}_\omega(\mathbb{R})$ , there exist  $T = T(\|f\|_{\mathcal{A}_\omega(\mathbb{R})}) > 0$  and unique  $u \in C([-T, T]; \mathcal{A}_\omega(\mathbb{R}))$  satisfying the following Duhamel formulation of (1.1):*

$$u(t) = S(t)f - i \int_0^t S(t-t') \mathcal{N}(u)(t') dt', \quad (1.10)$$

where  $S(t) = e^{it\partial_x^2}$ . Moreover, the solution map :  $f \in \mathcal{A}_\omega(\mathbb{R}) \mapsto u(t) \in \mathcal{A}_\omega(\mathbb{R})$  is locally Lipschitz continuous.

Our solution  $u(t)$  lies in  $\mathcal{A}_\omega(\mathbb{R})$  for all  $t \in [-T, T]$ . In particular,  $u(t)$  is almost periodic in  $x$  for all  $t \in [-T, T]$ . Moreover, it satisfies (1.1) in the distributional sense. See Lemmata 2.2 and 2.3 below.

In Section 2, we define the meaning of the linear propagator  $S(t) = e^{it\partial_x^2}$  in the almost periodic setting and discuss different properties of solutions to the homogeneous and non-homogeneous linear Schrödinger equations in the almost periodic setting. Then, we present the proof of Theorem 1.7, based on a simple fixed point argument. Since our approach makes use of the Fourier coefficients of functions in  $\mathcal{A}_\omega(\mathbb{R})$ , it is essential that the Fourier series associated to a function in  $\mathcal{A}_\omega(\mathbb{R})$  actually converges to it. See Lemma 2.1.

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<sup>2</sup>Obviously, an almost periodic function is periodic if and only if it has an integral basis consisting of a single element  $\omega \in \mathbb{R}$ .

**Remark 1.8.** Previously, Tsugawa [14] proved local well-posedness of the Korteweg-de Vries equation (KdV) on  $\mathbb{R}$ :

$$\partial_t u + \partial_x^3 u = u \partial_x u \quad (1.11)$$

with quasi-periodic initial data under some regularity condition. For fixed  $\boldsymbol{\omega} = \{\omega_j\}_{j=1}^N \in \mathbb{R}^N$  for some finite  $N \in \mathbb{N}$ , consider a quasi-periodic function  $f$  of the form:

$$f(x) = \sum_{\mathbf{n} \in \mathbb{Z}^N} \widehat{f}(\boldsymbol{\omega} \cdot \mathbf{n}) e^{i(\boldsymbol{\omega} \cdot \mathbf{n})x}. \quad (1.12)$$

Defining a Sobolev-type space<sup>3</sup>  $\mathcal{H}_{\boldsymbol{\omega}}^{\mathbf{s}}(\mathbb{R})$  for  $\mathbf{s} = \{s_j\}_{j=1}^N \in \mathbb{R}^N$  by the norm

$$\|f\|_{\mathcal{H}_{\boldsymbol{\omega}}^{\mathbf{s}}(\mathbb{R})} := \|\langle \mathbf{n} \rangle^{\mathbf{s}} \widehat{f}(\boldsymbol{\omega} \cdot \mathbf{n})\|_{\ell_{\mathbf{n}}^2(\mathbb{Z}^N)}, \quad \langle \mathbf{n} \rangle^{\mathbf{s}} := \prod_{j=1}^N (1 + |n_j|^2)^{\frac{s_j}{2}}, \quad (1.13)$$

it follows from Lemma 2.2 (i) in [14] that NLS (1.1) is locally well-posed in  $\mathcal{H}_{\boldsymbol{\omega}}^{\mathbf{s}}(\mathbb{R})$  as long as  $\min(s_1, \dots, s_N) > \frac{1}{2}$ . In this case, we have  $\mathcal{A}_{\boldsymbol{\omega}}(\mathbb{R}) \supset \mathcal{H}_{\boldsymbol{\omega}}^{\mathbf{s}}(\mathbb{R})$  by Cauchy-Schwarz inequality, and thus Theorem 1.7 extends this local well-posedness result of (1.1) with quasi-periodic initial data implied by Lemma 2.2 (i) in [14]. It is not clear if there is a natural way to define a Sobolev-type space analogous to (1.13) in the almost periodic setting, which guarantees that every function in the space can be represented by its Fourier series.

In view of Theorem 1.7, it is natural to consider the global-in-time behavior of solutions to (1.1). This is, however, an extremely difficult question in general. Consider the following NLS with the standard power nonlinearity:

$$i\partial_t u + \partial_x^2 u = \pm |u|^{p-1} u. \quad (1.14)$$

A standard approach to construct global-in-time solutions is to use conservation laws. There are several (formal) conservation laws for (1.14), including the mass conservation  $Q(u) := \|u\|_{\mathcal{L}^2}^2 = M(|u|^2)$  and the ‘Hamiltonian’ conservation:

$$H(u) = \frac{1}{2} M(|\partial_x u|^2) \pm \frac{1}{p+1} M(|u|^{p+1}).$$

On the one hand, in order to make use of the mass conservation in constructing global-in-time solutions, one needs to prove local well-posedness in  $AP(\mathbb{R})$  endowed with the  $\mathcal{L}^2$ -norm. This seems to be beyond our current technology in the quasi- and almost periodic setting due to the lack of Strichartz estimates. Note that while every function  $f \in AP(\mathbb{R})$  has a finite  $\mathcal{L}^2$ -norm, (i)  $AP(\mathbb{R})$  is not complete with respect to the  $\mathcal{L}^2$ -norm and (ii) its Fourier series does not necessarily converges to  $f$ . Hence, in proceeding with Fourier analytic approach, it seems that one needs to work in a subclass, where functions are actually represented by their Fourier series. For example, see Lemma 1.5 above.

On the other hand, assuming that  $u$  is of the form (1.9), we formally have

$$M(|\partial_x u|^2) = \sum_{\mathbf{n} \in \mathbb{Z}^N} (\boldsymbol{\omega} \cdot \mathbf{n})^2 |\widehat{u}(\boldsymbol{\omega} \cdot \mathbf{n})|^2.$$

In general,  $(\boldsymbol{\omega} \cdot \mathbf{n})^2$  can be arbitrarily close to 0 and thus  $M(|\partial_x u|^2)$  (and hence the Hamiltonian) is not strong enough to control relevant norms for iterating a local argument.

<sup>3</sup>This Sobolev-type space  $\mathcal{H}^{\mathbf{s}}(\mathbb{R})$  is basically the space  $G^{\mathbf{s},0}$  defined in [14].

There are, however, several known global existence results for cubic NLS, (1.14) with  $p = 3$ , and KdV in the almost periodic and quasi-periodic setting. Note that the following results rely heavily on the inverse spectral method and on the complete integrability of the equations.<sup>4</sup> Egorova [7] and Boutet de Monvel-Egorova [3] constructed global-in-time solutions to KdV and cubic NLS with almost periodic initial data, assuming some conditions, including Cantor-like spectra for the corresponding Schrödinger operator (for KdV) and Dirac operator (for cubic NLS). In particular, the class of almost periodic initial data in [7, 3] includes almost periodic functions  $f$  that can be approximated by periodic functions  $f_n$  of growing periods  $\alpha_n \rightarrow \infty$  in a local Sobolev norm:  $\sup_{x \in \mathbb{R}} \|\cdot\|_{H^s([x, x+1])}$  with  $s \geq 4$  for KdV and  $s \geq 3$  for cubic NLS.<sup>5</sup> Moreover, convergence of  $f_n$  to  $f$  in this local Sobolev norm is assumed to be exponentially fast. It is worthwhile to mention that the solutions constructed in [7, 3] are almost periodic in both  $t$  and  $x$ . There is also a recent global well-posedness result of KdV with quasi-periodic initial data by Damanik-Goldstein [6]. Their result states that if the Fourier coefficient  $\hat{f}(\omega \cdot \mathbf{n})$  of a ‘small’ quasi-periodic initial condition  $f$  of the form (1.12) decays exponentially fast (in  $\mathbf{n}$ ) and a Diophantine condition on  $\omega$  is satisfied, then there exists a unique global solution whose Fourier coefficient also decays exponentially fast (with a slightly worse constant).

Another approach for constructing global solutions is to consider the problem for small initial data. Indeed, in the usual setting on  $\mathbb{R}^d$ , i.e. assuming that functions belong to the usual Lebesgue spaces  $L^q(\mathbb{R}^d)$  or Sobolev spaces  $H^s(\mathbb{R}^d)$ , we have small data global well-posedness and scattering for NLS (1.1) on  $\mathbb{R}^d$ , for example, for  $p_s < p < 1 + \frac{4}{d-2}$ , where  $p_s$  is the Strauss exponent given by

$$p_s = \frac{d + 2 + \sqrt{d^2 + 12d + 4}}{2d}.$$

The proof of this result relies on the decay of linear solutions on  $\mathbb{R}^d$ . See [4]. On the contrary, there is no such decay of linear solutions in the almost periodic setting. Hence, there seems to be no natural adaptation of small data global existence theory to the almost periodic setting.

Next, let us discuss finite time blowup solutions to (1.1) in the almost periodic setting. Since a periodic function is in particular almost periodic, known results on finite time blowup solutions in the periodic setting such as [10, 11] provide instances of finite time blowup results in the almost periodic setting (where initial data are periodic). There seems to be, however, no known result on finite time blowup solutions in a generic (i.e. non-periodic) almost periodic setting.

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<sup>4</sup> There is a special subclass of almost periodic functions called *limit periodic* functions, consisting of uniform limits of periodic functions. In our recent paper [12], we proved global well-posedness of the defocusing NLS with nonlinearity  $|u|^{2k}u$ ,  $k \in \mathbb{N}$ , with limit periodic functions as initial data under some regularity assumption. In particular, our proof does not rely on the complete integrability, even when  $k = 1$ . Moreover, when  $k \geq 2$ , it provides the first instance of global existence for the defocusing NLS with (a subclass of) almost periodic initial data that are not quasi-periodic.

<sup>5</sup>Note that when  $s = 0$ , this local Sobolev norm corresponds to Stepanov’s  $S^2$ -norm used for Stepanov’s generalized almost periodic functions.

In the following, we consider the Cauchy problem of the following NLS:

$$\begin{cases} i\partial_t u + \partial_x^2 u = \lambda |u|^p, \\ u|_{t=0} = f, \end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \quad (1.15)$$

for  $p \in 2\mathbb{N}$  and  $\lambda \in \mathbb{C}$ . Then, we have the following result on finite time blowup solutions in a generic almost periodic setting.

**Theorem 1.9.** *Let  $p \in 2\mathbb{N}$ . Fix  $\omega = \{\omega_j\}_{j=1}^\infty \in \mathbb{R}^\mathbb{N}$ . Let  $u \in C(I; \mathcal{A}_\omega(\mathbb{R}))$  be the solution to (1.15) with  $u|_{t=0} = f \in \mathcal{A}_\omega(\mathbb{R})$ , where  $I = (-T_-, T_+)$  is the maximal time interval of existence, containing  $t = 0$ . Suppose that  $\lambda \in \mathbb{C}$  and the mean value  $M(f)$  of  $f$ , defined in (1.2), satisfy*

$$\operatorname{Re} \lambda \cdot \operatorname{Im} M(f) \neq 0 \quad \text{or} \quad \operatorname{Im} \lambda \cdot \operatorname{Re} M(f) \neq 0. \quad (1.16)$$

*Then, we have  $\min(T_-, T_+) < \infty$ . Namely, the solution  $u$  blows up in a finite time, either forward or backward in time. More precisely, if (1.16) holds, then we have one of the following scenarios.*

(i) *Suppose that we have*

$$\operatorname{Re} \lambda \cdot \operatorname{Im} M(f) < 0 \quad \text{or} \quad \operatorname{Im} \lambda \cdot \operatorname{Re} M(f) > 0. \quad (1.17)$$

*Then, the forward maximal time  $T_+$  of existence of the solution  $u$  is finite and we have  $\liminf_{t \nearrow T_+} \|u(t)\|_{\mathcal{A}_\omega(\mathbb{R})} = \infty$ .*

(ii) *Suppose that we have*

$$\operatorname{Re} \lambda \cdot \operatorname{Im} M(f) > 0 \quad \text{or} \quad \operatorname{Im} \lambda \cdot \operatorname{Re} M(f) < 0. \quad (1.18)$$

*Then, the backward maximal time  $T_-$  of existence of the solution  $u$  is finite and we have  $\liminf_{t \searrow -T_-} \|u(t)\|_{\mathcal{A}_\omega(\mathbb{R})} = \infty$ .*

Previously, Ikeda-Wakasugi [8] and the author [11] obtained analogous results for (1.15) on  $\mathbb{R}^d$  (with  $1 < p \leq \frac{2}{d}$  for initial data in  $L^2(\mathbb{R}^d)$ ) and on  $\mathbb{T}^d$  (with  $p \in 2\mathbb{N}$ ). Theorem 1.9 can be viewed as an extension of the periodic result in [11] to the generic almost periodic setting. In particular, note that Theorem 1.9 holds (i) even for small initial data and (ii) even above the Strauss exponent, i.e.  $p > p_s$ , provided that (1.16) is satisfied. This is a sharp contrast with the usual Euclidean setting on  $\mathbb{R}$ , where we have small data global well-posedness when  $p > p_s$ . The proof of Theorem 1.9 follows the basic lines in [8, 11] but we need to proceed more carefully due to the almost periodic setting. As in the proof of Theorem 1.7, we make essential use of properties of functions in  $\mathcal{A}_\omega(\mathbb{R})$ . We present the proof of Theorem 1.9 in Section 3.

**Remark 1.10.** Suppose that (1.16) is satisfied. Then, it follows from Theorem 1.9 that any solution to (1.15) on  $[0, \infty)$  or  $(-\infty, 0]$  must satisfy a global space-time bound. For example, if  $u$  is a solution to (1.15) on  $[0, \infty)$ , then we have  $\operatorname{Re} \lambda \cdot \operatorname{Im} M(f) > 0$  and  $\operatorname{Im} \lambda \cdot \operatorname{Re} M(f) < 0$ . Then, we have

$$\int_0^\infty M(|u(t)|^p) dt < \infty.$$

Hence, in view of Lemma 1.3, any global solution on  $[0, \infty)$  must go to 0 as  $t \rightarrow \infty$  in some averaged sense. See Remark 3.4 for the proof.

**Remark 1.11.** The notion of almost periodic functions can be extended to higher dimensions.<sup>6</sup> One may extend the results in this paper to the higher dimensional setting. We, however, focus on the one-dimensional case for simplicity of the presentation.

## 2. LOCAL WELL-POSEDNESS

In this section, we present the proof of Theorem 1.7. Fix  $\omega = \{\omega_j\}_{j=1}^\infty \in \mathbb{R}^\mathbb{N}$  in the following.

**2.1. On the function space  $\mathcal{A}_\omega(\mathbb{R})$ .** We first go over some important properties of the space  $\mathcal{A}_\omega(\mathbb{R})$  of almost periodic functions with a common integral basis  $\omega$ .

**Lemma 2.1.** *Given  $f \in \mathcal{A}_\omega(\mathbb{R})$ , we have*

$$f(x) = \sum_{\mathbf{n} \in \mathbb{Z}^\mathbb{N}} \widehat{f}(\omega \cdot \mathbf{n}) e^{i(\omega \cdot \mathbf{n})x}. \quad (2.1)$$

*Namely,  $f$  is given by its Fourier series.<sup>7</sup> Moreover,  $\mathcal{A}_\omega(\mathbb{R})$  is a Banach algebra.*

*Proof.* Let  $\{r_j\}_{j=1}^\infty$  be an enumeration of  $\mathbb{Z}^\mathbb{N}$ . For  $N \in \mathbb{N}$ , we set  $B_N = \{r_j\}_{j=1}^N$ . Given  $f \in \mathcal{A}_\omega(\mathbb{R})$ , define a trigonometric polynomial  $f_N$  by

$$f_N(x) = \sum_{\mathbf{n} \in B_N} \widehat{f}(\omega \cdot \mathbf{n}) e^{i(\omega \cdot \mathbf{n})x}, \quad (2.2)$$

Then, by the Weierstrass  $M$ -test,  $f_N$  converges uniformly. Hence, by Lemma 1.4, we obtain (2.1). Note that  $f_N$  also converges to  $f$  in  $\mathcal{A}_\omega(\mathbb{R})$ .

Given a Cauchy sequence  $\{f_k\}_{k=1}^\infty$  in  $\mathcal{A}_\omega(\mathbb{R})$ , it follows from the completeness of  $\ell^1$ , that  $f_k$  converges to some function  $f$  defined by the Fourier series (2.1) with respect to the  $\mathcal{A}_\omega(\mathbb{R})$ -norm. We need to show that  $f \in AP(\mathbb{R})$ . Since  $\{f_k\}_{k=1}^\infty$  is Cauchy with respect to the  $\mathcal{A}_\omega(\mathbb{R})$ -norm, then it is Cauchy in  $L^\infty(\mathbb{R})$ . Noting that  $AP(\mathbb{R})$  is closed with respect to the  $L^\infty$ -norm, it follows that  $f_k$  converges some function in  $AP(\mathbb{R})$ . Hence, by uniqueness of a limit, we conclude that  $f \in AP(\mathbb{R})$ . This proves completeness of  $\mathcal{A}_\omega(\mathbb{R})$ .

Lastly, given  $f, g \in \mathcal{A}_\omega(\mathbb{R})$ , we have  $fg \in AP(\mathbb{R})$ . Moreover, we have  $\sigma(fg) \subset \omega \cdot \mathbb{Z}^\mathbb{N}$  and

$$\widehat{fg}(\omega \cdot \mathbf{n}) = \sum_{\mathbf{m} \in \mathbb{Z}^\mathbb{N}} \widehat{f}(\omega \cdot (\mathbf{n} - \mathbf{m})) \widehat{g}(\omega \cdot \mathbf{m}).$$

Then, by Young's inequality, we have

$$\|fg\|_{\mathcal{A}_\omega(\mathbb{R})} \leq \|f\|_{\mathcal{A}_\omega(\mathbb{R})} \|g\|_{\mathcal{A}_\omega(\mathbb{R})}. \quad (2.3)$$

Namely,  $\mathcal{A}_\omega(\mathbb{R})$  is an algebra. □

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<sup>6</sup> Note that almost periodic functions on  $\mathbb{R}^d$  are almost periodic in each variable, but the converse is not true.

<sup>7</sup> If  $\omega$  is not linearly independent, the right-hand side of (2.1) may not represent the Fourier series associated to  $f$ . Nonetheless, the claim in Lemma 2.1 holds. See Remark 1.6.



**2.2. On the linear Schrödinger equation.** In this subsection, we study the properties of solutions to the homogeneous and nonhomogeneous Schrödinger equations in the almost periodic setting. We first consider the homogeneous linear Schrödinger equation:

$$\begin{cases} i\partial_t u + \partial_x^2 u = 0, \\ u|_{t=0} = f \in \mathcal{A}_\omega(\mathbb{R}), \end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}. \quad (2.4)$$

Given  $f \in \mathcal{A}_\omega(\mathbb{R})$  satisfying (2.1), we define the linear propagator  $S(t) = e^{it\partial_x^2}$  by

$$S(t)f := \sum_{\mathbf{n} \in \mathbb{Z}^N} \widehat{f}(\omega \cdot \mathbf{n}) e^{-i(\omega \cdot \mathbf{n})^2 t} e^{i(\omega \cdot \mathbf{n})x}. \quad (2.5)$$

**Lemma 2.2.** *Given  $f \in \mathcal{A}_\omega(\mathbb{R})$ , let  $u = S(t)f$  be as in (2.5).*

(i) *The function  $u = S(t)f$  satisfies (2.4) in the distributional sense. Namely, we have*

$$\iint_{\mathbb{R} \times \mathbb{R}} u \left( -i\partial_t \phi + \partial_x^2 \phi \right) dx dt = 0, \quad (2.6)$$

*for any test function  $\phi \in C_c^\infty(\mathbb{R}_t \times \mathbb{R}_x)$ .*

(ii) *The function  $u = S(t)f$  lies in  $C(\mathbb{R}; \mathcal{A}_\omega(\mathbb{R}))$ . Moreover, we have*

$$\|S(t)f\|_{C(\mathbb{R}; \mathcal{A}_\omega(\mathbb{R}))} \leq \|f\|_{\mathcal{A}_\omega(\mathbb{R})}. \quad (2.7)$$

(iii) *The function  $u = S(t)f$  is almost periodic in both  $t$  and  $x$ . Moreover, we have  $u \in AP(\mathbb{R}; \mathcal{A}_\omega(\mathbb{R}))$ . Namely, the function  $u = S(t)f$  is almost periodic in  $t$  with values in  $\mathcal{A}_\omega(\mathbb{R})$ .*

In the following, we do not make use of Lemma 2.2 (iii). We, however, decided to include it due to its independent interest. The same comment applies to Lemma 2.3 (iii) below.

*Proof.* (iii) Given an enumeration  $\{r_j\}_{j=1}^\infty$  of  $\mathbb{Z}^N$ , let  $B_N = \{r_j\}_{j=1}^N$ ,  $N \in \mathbb{N}$ . Define  $f_N$  as in (2.2) and  $u_N$  by

$$u_N(t) = S(t)f_N := \sum_{\mathbf{n} \in B_N} \widehat{f}(\omega \cdot \mathbf{n}) e^{-i(\omega \cdot \mathbf{n})^2 t} e^{i(\omega \cdot \mathbf{n})x}. \quad (2.8)$$

By writing  $u_N = \sum_{\mathbf{n} \in B_N} c_N e^{-i(\omega \cdot \mathbf{n})^2 t}$  with  $c_N = \widehat{f}(\omega \cdot \mathbf{n}) e^{i(\omega \cdot \mathbf{n})x} \in \mathcal{A}_\omega(\mathbb{R})$ , we see that  $u_N$  is a trigonometric polynomial with values in the Banach space  $\mathcal{A}_\omega(\mathbb{R})$ . Moreover, given  $\varepsilon > 0$ , there exists  $N_0 \in \mathbb{N}$  such that

$$\sup_{t \in \mathbb{R}} \|u(t) - u_N(t)\|_{\mathcal{A}_\omega(\mathbb{R})} = \|f - f_N\|_{\mathcal{A}_\omega(\mathbb{R})} = \sum_{\mathbf{n} \in \mathbb{Z}^N \setminus B_N} |\widehat{f}(\omega \cdot \mathbf{n})| < \varepsilon \quad (2.9)$$

for all  $N \geq N_0$ . Namely,  $u = S(t)f$  is uniformly approximated by the trigonometric polynomials  $u_N$ . Therefore,  $u$  is almost periodic in  $t$  with values in  $\mathcal{A}_\omega(\mathbb{R})$ . Since  $\mathcal{A}_\omega(\mathbb{R}) \subset L^\infty(\mathbb{R})$ , (2.9) implies that  $u_N$  converges to  $u$  in  $L_{t,x}^\infty(\mathbb{R} \times \mathbb{R})$ . Noting that  $u_N$  is a trigonometric polynomial in  $t$  and  $x$ , uniformly converging to  $u$ , the first claim in (iii) follows.

(i) Since the sum in (2.8) is over a finite set of indices, we see that  $u_N$  is a smooth solution to (2.4) in the classical sense, where the initial condition is replaced by  $f_N$ . In particular, we have

$$\iint_{\mathbb{R} \times \mathbb{R}} u_N \left( -i\partial_t \phi + \partial_x^2 \phi \right) dx dt = 0, \quad (2.10)$$

for any test function  $\phi \in C_c^\infty(\mathbb{R}_t \times \mathbb{R}_x)$ . Then, we have

$$\begin{aligned} & \left| \iint_{\mathbb{R} \times \mathbb{R}} (u - u_N) \left( -i\partial_t \phi + \partial_x^2 \phi \right) dx dt \right| \\ & \leq \|u - u_N\|_{L_{t,x}^\infty(\mathbb{R} \times \mathbb{R})} \left( \|\phi\|_{W_t^{1,1} L_x^1} + \|\phi\|_{L_t^1 W_x^{2,1}} \right) \longrightarrow 0, \end{aligned} \quad (2.11)$$

as  $N \rightarrow \infty$ . Hence, (2.6) follows from (2.10) and (2.11).

(ii) By (iii), we have  $AP(\mathbb{R}; \mathcal{A}_\omega(\mathbb{R}))$ . Hence, the first claim in (ii) follows since an almost periodic function is continuous. Lastly, note that (2.7) follows from (2.5) and the definition of the  $\mathcal{A}_\omega(\mathbb{R})$ -norm. This completes the proof of Lemma 2.2.  $\square$

Next, we consider the nonhomogeneous linear Schrödinger equation:

$$\begin{cases} i\partial_t u + \partial_x^2 u = F, \\ u|_{t=0} = 0, \end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \quad (2.12)$$

for  $F \in L^\infty([-T, T]; \mathcal{A}_\omega(\mathbb{R}))$ ,  $T > 0$ . Then, the solution  $u$  to (2.12) is formally given by

$$\begin{aligned} u(t) &:= -i \int_0^t S(t-t') F(t') dt' \\ &= -i \int_0^t \sum_{\mathbf{n} \in \mathbb{Z}^N} \widehat{F}(t', \boldsymbol{\omega} \cdot \mathbf{n}) e^{-i(\boldsymbol{\omega} \cdot \mathbf{n})^2(t-t')} e^{i(\boldsymbol{\omega} \cdot \mathbf{n})x} dt'. \end{aligned} \quad (2.13)$$

**Lemma 2.3.** *Let  $T > 0$ . Given  $F \in L^\infty([-T, T]; \mathcal{A}_\omega(\mathbb{R}))$ , let  $u$  be as in (2.13).*

(i) *The function  $u$  satisfies (2.12) in the distributional sense. Namely, we have*

$$\iint_{[-T, T] \times \mathbb{R}} u \left( -i\partial_t \phi + \partial_x^2 \phi \right) dx dt = \iint_{[-T, T] \times \mathbb{R}} F \phi dx dt, \quad (2.14)$$

for any test function  $\phi \in C_c^\infty([-T, T] \times \mathbb{R})$ .

(ii) *The function  $u$  defined in (2.13) lies in  $C([-T, T]; \mathcal{A}_\omega(\mathbb{R}))$ .<sup>8</sup> Moreover, we have*

$$\left\| \int_0^t S(t-t') F(t') dt' \right\|_{C([-T, T]; \mathcal{A}_\omega(\mathbb{R}))} \leq T \|F\|_{L^\infty([-T, T]; \mathcal{A}_\omega(\mathbb{R}))}. \quad (2.15)$$

(iii) *Let  $T = \infty$ . Suppose that  $F(t)$  is almost periodic in  $t$  with values in  $\mathcal{A}_\omega(\mathbb{R})$  whose Fourier series is given by*

$$F(t) \sim \sum_{j=1}^{\infty} c_j e^{i\lambda_j t}, \quad c_j \in \mathcal{A}_\omega(\mathbb{R}), \quad (2.16)$$

such that  $\{c_j\}_{j=1}^\infty \in \ell^1(\mathbb{N}; \mathcal{A}_\omega(\mathbb{R}))$ . In addition, assume that the following non-resonance condition holds:

$$\inf_{j \in \mathbb{N}} \inf_{\mathbf{n} \in \mathbb{Z}^N} |\lambda_j + (\boldsymbol{\omega} \cdot \mathbf{n})^2| > \delta > 0 \quad (2.17)$$

for some  $\delta > 0$ . Then,  $u$  defined in (2.13) is almost periodic in  $t$  and  $x$ . Moreover, we have  $u \in AP(\mathbb{R}; \mathcal{A}_\omega(\mathbb{R}))$ . Namely, it is almost periodic in  $t$  with values in  $\mathcal{A}_\omega(\mathbb{R})$ .

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<sup>8</sup>Indeed, it follows from the proof that  $u$  is uniformly continuous on  $[-T, T]$ ,  $T < \infty$ , with values in  $\mathcal{A}_\omega(\mathbb{R})$ .

*Proof.* (i) Since  $F \in L^\infty([-T, T]; \mathcal{A}_\omega(\mathbb{R}))$ , we have

$$F(t, x) = \sum_{\mathbf{n} \in \mathbb{Z}^N} \widehat{F}(t, \omega \cdot \mathbf{n}) e^{i(\omega \cdot \mathbf{n})x}, \quad (2.18)$$

for almost every  $t \in [-T, T]$ . As before, given an enumeration  $\{r_j\}_{j=1}^\infty$  of  $\mathbb{Z}^N$ , we define  $F_N$  by

$$F_N(t, x) = \sum_{\mathbf{n} \in B_N} \widehat{F}(t, \omega \cdot \mathbf{n}) e^{i(\omega \cdot \mathbf{n})x}, \quad (2.19)$$

for  $t \in [-T, T]$  such that (2.18) holds, where  $B_N = \{r_j\}_{j=1}^N$ ,  $N \in \mathbb{N}$ . In the following, by setting  $F(t) = 0$  on the exceptional set of measure 0 in  $[-T, T]$ , we simply assume that (2.18) and (2.19) hold for all  $t \in [-T, T]$ .

Now, define  $u_N$  by

$$u_N(t) := -i \int_0^t S(t-t') F_N(t') dt' = -i \sum_{\mathbf{n} \in B_N} \int_0^t \widehat{F}(t', \omega \cdot \mathbf{n}) e^{-i(\omega \cdot \mathbf{n})^2(t-t')} dt' e^{i(\omega \cdot \mathbf{n})x}. \quad (2.20)$$

Since  $\widehat{F}(t, \omega \cdot \mathbf{n}) \in L^\infty([-T, T]) \subset L^1([-T, T])$ , we see that  $u_N$  is absolutely continuous in  $t$  and smooth in  $x$ . Then, it is easy to see that such  $u_N$  satisfies (2.12) with  $F$  replaced by  $F_N$  for almost every  $t \in [-T, T]$  and every  $x \in \mathbb{R}$ . In particular, we have

$$\iint_{[-T, T] \times \mathbb{R}} u_N \left( -i \partial_t \phi + \partial_x^2 \phi \right) dx dt = \iint_{[-T, T] \times \mathbb{R}} F_N \phi dx dt \quad (2.21)$$

for any test function  $\phi \in C_c^\infty([-T, T] \times \mathbb{R})$ .

Note that  $S(t-t') F_N(t')$  is given by

$$S(t-t') F_N(t') = \sum_{\mathbf{n} \in B_N} \widehat{F}(t', \omega \cdot \mathbf{n}) e^{-i(\omega \cdot \mathbf{n})^2(t-t')} e^{i(\omega \cdot \mathbf{n})x}.$$

Then, we see that  $S(t-t') F_N(t')$  converges to  $S(t-t') F(t')$  in  $\mathcal{A}_\omega(\mathbb{R})$  for each fixed  $t' \in [-T, T]$  (and  $t \in \mathbb{R}$ ). Moreover, we have  $\sup_N \|S(t-t') F_N(t')\|_{\mathcal{A}_\omega(\mathbb{R})} \leq \|F(t')\|_{\mathcal{A}_\omega(\mathbb{R})} \in L^1_\nu([-T, T])$ . Hence, it follows from Dominated Convergence Theorem that  $u_N(t, x)$  converges to  $u(t, x)$  for every  $(t, x) \in [-T, T] \times \mathbb{R}$  and we have

$$u(t) = -i \sum_{\mathbf{n} \in \mathbb{Z}^N} \int_0^t \widehat{F}(t', \omega \cdot \mathbf{n}) e^{-i(\omega \cdot \mathbf{n})^2(t-t')} dt' e^{i(\omega \cdot \mathbf{n})x}. \quad (2.22)$$

Also, note that, for each  $t \in [-T, T]$ ,  $F_N(t)$  converges to  $F(t)$  in  $\mathcal{A}_\omega(\mathbb{R})$ . In particular,  $F_N(t, x)$  converges to  $F(t, x)$  for every  $(t, x) \in [-T, T] \times \mathbb{R}$ . Moreover, we have

$$|F_N(t, x)| \leq \|F\|_{L^\infty([-T, T]; \mathcal{A}_\omega(\mathbb{R}))} \quad \text{and} \quad |u_N(t, x)| \leq T \|F\|_{L^\infty([-T, T]; \mathcal{A}_\omega(\mathbb{R}))}$$

for all  $(t, x) \in [-T, T] \times \mathbb{R}$ . Therefore, by Dominated Convergence Theorem applied to both sides of (2.21), we obtain (2.14).

(ii) Note that  $\sum_{\mathbf{n} \in \mathbb{Z}^N \setminus B_N} |\widehat{F}(t, \omega \cdot \mathbf{n})|$  converges to 0 for each  $t \in [-T, T]$  as  $N \rightarrow \infty$  and that  $\sum_{\mathbf{n} \in \mathbb{Z}^N \setminus B_N} |\widehat{F}(t, \omega \cdot \mathbf{n})| \leq \|F(t)\|_{\mathcal{A}_\omega(\mathbb{R})} \in L^1([-T, T])$ . Then, by Dominated Convergence Theorem, we have

$$\lim_{N \rightarrow \infty} \int_{-T}^T \sum_{\mathbf{n} \in \mathbb{Z}^N \setminus B_N} |\widehat{F}(t, \omega \cdot \mathbf{n})| = 0. \quad (2.23)$$

Hence, it follows from (2.20), (2.22), and (2.23) that  $u_N$  converges to  $u$  in  $L^\infty([-T, T]; \mathcal{A}_\omega(\mathbb{R}))$ .

Fix  $t \in [-T, T]$  and  $\varepsilon > 0$ . Then, there exists  $N_0 \in \mathbb{N}$  sufficiently large such that

$$\|u_{N_0} - u\|_{L^\infty([-T, T]; \mathcal{A}_\omega(\mathbb{R}))} < \frac{\varepsilon}{4}. \quad (2.24)$$

Also, from (2.20) and Mean Value Theorem, there exists  $\delta_0 > 0$  such that

$$\begin{aligned} \|u_{N_0}(t + \delta) - u_{N_0}(t)\|_{\mathcal{A}_\omega(\mathbb{R})} &\leq \int_t^{t+\delta} \sum_{\mathbf{n} \in B_{N_0}} |\widehat{F}(t', \boldsymbol{\omega} \cdot \mathbf{n})| dt' \\ &\quad + \int_0^t \sum_{\mathbf{n} \in B_{N_0}} |\widehat{F}(t', \boldsymbol{\omega} \cdot \mathbf{n})| (e^{-i(\boldsymbol{\omega} \cdot \mathbf{n})^2(t+\delta)} - e^{-i(\boldsymbol{\omega} \cdot \mathbf{n})^2 t}) dt' \\ &\leq |\delta| \left(1 + T \max_{\mathbf{n} \in B_{N_0}} (\boldsymbol{\omega} \cdot \mathbf{n})^2\right) \|F\|_{L^\infty([-T, T]; \mathcal{A}_\omega(\mathbb{R}))} < \frac{\varepsilon}{2} \end{aligned} \quad (2.25)$$

for all  $|\delta| < \delta_0$  such that  $t + \delta \in [-T, T]$ . Therefore, from (2.24) and (2.25), we have

$$\|u(t + \delta) - u(t)\|_{\mathcal{A}_\omega(\mathbb{R})} \leq \|u_{N_0}(t + \delta) - u_{N_0}(t)\|_{\mathcal{A}_\omega(\mathbb{R})} + \frac{\varepsilon}{2} < \varepsilon$$

for all  $|\delta| < \delta_0$  such that  $t + \delta \in [-T, T]$ . This shows that  $u \in C(\mathbb{R}; \mathcal{A}_\omega(\mathbb{R}))$ . Lastly, note that (2.15) follows from (2.22) and the definition of the  $\mathcal{A}_\omega(\mathbb{R})$ -norm.

(iii) By assumption on  $c_j$  and Lemma 2.1, the Fourier series (in  $x$ ) associated to  $c_j$  converges uniformly to  $c_j$  and we have

$$c_j(x) = \sum_{\mathbf{n} \in \mathbb{Z}^N} \widehat{c}_j(\boldsymbol{\omega} \cdot \mathbf{n}) e^{i(\boldsymbol{\omega} \cdot \mathbf{n})x}, \quad (2.26)$$

for all  $j \in \mathbb{N}$ . Given  $\varepsilon > 0$ , choose  $J_0 \in \mathbb{N}$  such that

$$\sup_{t \in \mathbb{R}} \left\| \sum_{j=J_0}^{\infty} c_j e^{i\lambda_j t} \right\|_{\mathcal{A}_\omega(\mathbb{R})} \leq \sum_{j=J_0}^{\infty} \|c_j\|_{\mathcal{A}_\omega(\mathbb{R})} < \varepsilon.$$

In particular, the Fourier series (in  $t$ ) associated to  $F$  converges in  $\mathcal{A}_\omega(\mathbb{R})$  uniformly in  $t$ . Hence, by Lemma 1.4, they must converge to  $F$ . See also Theorem 6.14 in [5]. Then, from (2.26), we have

$$F(t, x) = \sum_{j=1}^{\infty} c_j(x) e^{i\lambda_j t} = \sum_{j=1}^{\infty} \sum_{\mathbf{n} \in \mathbb{Z}^N} \widehat{c}_j(\boldsymbol{\omega} \cdot \mathbf{n}) e^{i(\boldsymbol{\omega} \cdot \mathbf{n})x} e^{i\lambda_j t}. \quad (2.27)$$

Note that the series on the right-hand side of (2.27) converges absolutely and uniformly in  $t$  and  $x$ . Comparing (2.19) and (2.27), we conclude that

$$\widehat{F}(t, \boldsymbol{\omega} \cdot \mathbf{n}) = \sum_{j=1}^{\infty} \widehat{c}_j(\boldsymbol{\omega} \cdot \mathbf{n}) e^{i\lambda_j t}. \quad (2.28)$$

Then, from (2.22) and (2.28) with Dominated Convergence Theorem, we have

$$\begin{aligned} u(t) &= -i \sum_{\mathbf{n} \in \mathbb{Z}^N} \int_0^t \sum_{j=1}^{\infty} \widehat{c}_j(\boldsymbol{\omega} \cdot \mathbf{n}) e^{i(\lambda_j + (\boldsymbol{\omega} \cdot \mathbf{n})^2)t'} dt' e^{-i(\boldsymbol{\omega} \cdot \mathbf{n})^2 t} e^{i(\boldsymbol{\omega} \cdot \mathbf{n})x} \\ &= \sum_{\mathbf{n} \in \mathbb{Z}^N} \sum_{j=1}^{\infty} \widehat{c}_j(\boldsymbol{\omega} \cdot \mathbf{n}) \frac{e^{i(\boldsymbol{\omega} \cdot \mathbf{n})^2 t} - e^{i\lambda_j t}}{\lambda_j + (\boldsymbol{\omega} \cdot \mathbf{n})^2} e^{i(\boldsymbol{\omega} \cdot \mathbf{n})x}. \end{aligned} \quad (2.29)$$

In view of the non-resonance condition (2.17), the double series on the right-hand side of (2.29) converges absolutely and uniformly in  $t$  and  $x$ . This proves almost periodicity of  $u$  in both  $t$  and  $x$ . Moreover, noting that  $\widehat{c}_j(\boldsymbol{\omega} \cdot \mathbf{n}) e^{i(\boldsymbol{\omega} \cdot \mathbf{n})x} \in \mathcal{A}_{\boldsymbol{\omega}}(\mathbb{R})$  with the norm given by  $|\widehat{c}_j(\boldsymbol{\omega} \cdot \mathbf{n})|$  for each  $j \in \mathbb{N}$  and  $\mathbf{n} \in \mathbb{Z}^N$ , we see that  $u$  is a limit of  $\mathcal{A}_{\boldsymbol{\omega}}(\mathbb{R})$ -valued trigonometric polynomials, uniformly in  $t$ , i.e.  $u$  has the approximation property. Therefore,  $u$  is almost periodic in  $t$  with values in  $\mathcal{A}_{\boldsymbol{\omega}}(\mathbb{R})$ .  $\square$

**2.3. Fixed point argument.** Finally, we are ready to prove Theorem 1.7. Given  $f \in \mathcal{A}_{\boldsymbol{\omega}}(\mathbb{R})$ , define  $\Gamma = \Gamma_f$  by

$$\Gamma u(t) = \Gamma_f u(t) := S(t)f - i \int_0^t S(t-t') \mathcal{N}(u)(t') dt'. \quad (2.30)$$

We show that  $\Gamma$  is a contraction on

$$B = \{u \in C([-T, T]; \mathcal{A}_{\boldsymbol{\omega}}(\mathbb{R})) : \|u\|_{C([-T, T]; \mathcal{A}_{\boldsymbol{\omega}}(\mathbb{R}))} \leq 2\|f\|_{\mathcal{A}_{\boldsymbol{\omega}}(\mathbb{R})}\}$$

for some  $T > 0$ . Indeed, by (2.3) and Lemmata 2.2 and 2.3, we have

$$\|u\|_{C([-T, T]; \mathcal{A}_{\boldsymbol{\omega}}(\mathbb{R}))} \leq \|f\|_{\mathcal{A}_{\boldsymbol{\omega}}(\mathbb{R})} + T\|u\|_{C([-T, T]; \mathcal{A}_{\boldsymbol{\omega}}(\mathbb{R}))}^p,$$

and

$$\|u - v\|_{C([-T, T]; \mathcal{A}_{\boldsymbol{\omega}}(\mathbb{R}))} \leq CT \left( \|u\|_{C([0, T]; \mathcal{A}_{\boldsymbol{\omega}}(\mathbb{R}))}^{p-1} + \|v\|_{C([-T, T]; \mathcal{A}_{\boldsymbol{\omega}}(\mathbb{R}))}^{p-1} \right) \|u - v\|_{C([-T, T]; \mathcal{A}_{\boldsymbol{\omega}}(\mathbb{R}))}$$

for  $u, v \in B$ . Therefore, we conclude that  $\Gamma$  is a contraction on  $B$  as long as

$$T < \frac{1}{2^{p+1} C \|f\|_{\mathcal{A}_{\boldsymbol{\omega}}(\mathbb{R})}^{p-1}}.$$

The Lipschitz continuity of the solution map on  $\mathcal{A}_{\boldsymbol{\omega}}(\mathbb{R})$  follows from a similar argument.

### 3. FINITE TIME BLOWUP SOLUTIONS

In this section, we present the proof of Theorem 1.9. We only work on the positive time intervals  $[0, T)$ ,  $0 < T \leq \infty$ , and prove Theorem 1.9 (i) under (1.17). Note that Theorem 1.9 (ii) easily follows from Theorem 1.9 (i). Given a solution  $u$  to (1.15) on  $(-T, 0] \times \mathbb{R}$ , the function  $v(t, x) := u(-t, x)$  satisfies  $i\partial_t v - \partial_x^2 v = (-\lambda)|v|^p$  on  $[0, T) \times \mathbb{R}$ . Then, it suffices to note that the sign change in front of  $\partial_x^2 v$  does not affect the proof of Theorem 1.9 (i). In the following, fix  $p \in 2\mathbb{N}$ . and  $\boldsymbol{\omega} = \{\omega_j\}_{j=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ .

First, let us define the notion of weak solutions as in [8, 11].

**Definition 3.1.** Let  $T > 0$ . We say that  $u$  is a local weak solution to (1.15) on  $[0, T)$  with initial condition  $u|_{t=0} = f \in AP(\mathbb{R})$  if  $u \in L^\infty([0, T); \mathcal{A}_{\boldsymbol{\omega}}(\mathbb{R}))$  and

$$\int_0^T \int_{\mathbb{R}} u \left( -i\partial_t \phi + \partial_x^2 \phi \right) dx dt = i \int_{\mathbb{R}} f(x) \phi(0, x) dx + \lambda \int_0^T \int_{\mathbb{R}} |u|^p \phi dx dt, \quad (3.1)$$

for any test function  $\phi \in C_c^\infty((-\infty, T) \times \mathbb{R})$ . If  $T > 0$  can be made arbitrarily large, then we say that  $u$  is a global weak solution on  $[0, \infty)$ .

We now present two important propositions for proving Theorem 1.9.

**Proposition 3.2.** *Assume (1.17). If  $u$  is a global-in-time weak solution to (1.15) on  $[0, \infty)$ , then  $u(t) = 0$  for almost every  $t \in [0, \infty)$ .*

Our solution constructed in Theorem 1.7 satisfies the following Duhamel formulation:

$$u(t) = S(t)f - i\lambda \int_0^t S(t-t')|u(t')|^p dt'. \quad (3.2)$$

The next proposition guarantees that our solution  $u$  in Theorem 1.7 indeed satisfies the weak formulation (3.1).

**Proposition 3.3.** *Given  $f \in \mathcal{A}_\omega(\mathbb{R})$ , if  $u \in C([0, T]; \mathcal{A}_\omega(\mathbb{R}))$  satisfies the Duhamel formulation (3.2) on  $[0, T)$  for some  $T > 0$ , then it is a weak solution to (1.15) on  $[0, T)$  in the sense of Definition 3.1.*

We first prove Theorem 1.9 using Propositions 3.2 and 3.3. Then, we present the proofs of Propositions 3.2 and 3.3.

Let  $u \in C([0, T_+); \mathcal{A}_\omega(\mathbb{R}))$  be the solution to (1.15) with  $u|_{t=0} = f \in \mathcal{A}_\omega(\mathbb{R})$ , where  $[0, T_+)$  is the forward maximal time interval of existence. Suppose that (1.17) holds. On the one hand, this implies  $f \neq 0$ . Then, it follows from uniqueness in Theorem 1.7 that  $u \not\equiv 0$ . On the other hand, from Proposition 3.2 and the continuity of  $u$  in time, we conclude that  $u \equiv 0$ . This is a contradiction. Therefore, the solution  $u$  can not be global on  $[0, \infty)$  and must blow up at some finite time  $T_+ > 0$ .

Theorem 1.7 guarantees the following blowup alternative; if  $u$  is a solution in  $C([0, T]; \mathcal{A}_\omega(\mathbb{R}))$ , then either (a) there exists  $\varepsilon > 0$  such that  $u$  can be extended to  $[0, T + \varepsilon)$  or (b)  $\liminf_{t \nearrow T} \|u(t)\|_{\mathcal{A}_\omega(\mathbb{R})} = \infty$ . Since  $T_+ < \infty$ , we conclude that  $\liminf_{t \nearrow T_+} \|u(t)\|_{\mathcal{A}_\omega(\mathbb{R})} = \infty$ . This completes the proof of Theorem 1.9.

In the remaining part of this section, we present the proofs of Propositions 3.2 and Proposition 3.3.

*Proof of Proposition 3.2.* The proof is based on the test-function method by Zhang [15]. While we closely follows the arguments in [8, 11], some care must be taken due to the almost periodic nature of the problem. In particular, while there is only one parameter for the space-time cutoff functions in [8, 11], we need to introduce a space-time cutoff function depending on two parameters. This is mainly due to the fact that the  $L^p$ -norm of a non-zero almost periodic function is infinite for any finite value of  $p$  and thus we need to use the mean value  $M(|u|^p)$  of  $|u|^p$  instead. For simplicity of presentation, we only prove Proposition 3.2 when  $\operatorname{Re} \lambda \cdot \operatorname{Im} M(f) < 0$ . Without loss of generality, assume  $\operatorname{Re} \lambda > 0$  and  $\operatorname{Im} M(f) < 0$ .

Let  $\theta : [0, \infty) \rightarrow [0, 1]$  be a smooth cutoff function supported on  $[0, 1)$  such that  $\theta \equiv 1$  on  $[0, \frac{1}{2})$  and  $|\theta'(t)|, |\theta''(t)| \leq C$  for all  $t \in [0, \infty)$ . Moreover, we impose that

$$\frac{|\theta'(t)|^2}{\theta(t)} \leq C \quad (3.3)$$

for all  $t \in [0, 1]$ . Fix  $\varepsilon \in (0, 1)$ . Given  $L > 1$ , we define a smooth cutoff function  $\eta_L : \mathbb{R} \rightarrow [0, 1]$  by

$$\eta_L(x) = \begin{cases} 1 & \text{for } |x| \leq L - L^{1-\varepsilon}, \\ \theta\left(\frac{|x| - L + L^{1-\varepsilon}}{L^{1-\varepsilon}}\right) & \text{for } |x| > L - L^{1-\varepsilon}. \end{cases}$$

In particular,  $\text{supp } \eta_L \subset [-L, L]$  and by (3.3), we have

$$|\partial_x^2 \eta_L(x)|, \frac{|\partial_x \eta_L(x)|^2}{\eta_L(x)} \leq \frac{C}{L^{2-2\varepsilon}} \quad (3.4)$$

on  $(-L, L)$ . Finally, given  $R, L > 1$ , we define the space-time cutoff function  $\phi_{R,L}$  by  $\phi_{R,L}(t, x) = \theta_R(t) \cdot \eta_L(x)$ , where  $\theta_R(t) := \theta(R^{-2}t)$ .

For  $R, L > 1$ , define  $I_{R,L}$  by

$$I_{R,L} := \text{Re } \lambda \cdot \frac{1}{2L} \int_0^{R^2} \int_{-L}^L |u|^p \phi_{R,L}^{p'} dx dt,$$

where  $p'$  is the Hölder conjugate exponent of  $p$ . Since  $\text{Im } M(f) < 0$  and  $AP(\mathbb{R}) \subset L^\infty(\mathbb{R})$ , we have

$$\begin{aligned} & \lim_{L \rightarrow \infty} \text{Im} \left[ \frac{1}{2L} \int_{-L}^L f(x) \eta_L^{p'}(x) dx \right] \\ & \leq \lim_{L \rightarrow \infty} \frac{L - L^{1-\varepsilon}}{L} \text{Im} \left[ \frac{1}{2(L - L^{1-\varepsilon})} \int_{-L+L^{1-\varepsilon}}^{L-L^{1-\varepsilon}} f(x) dx \right] + \lim_{L \rightarrow \infty} \frac{C}{L^\varepsilon} \|f\|_{L^\infty(\mathbb{R})} \\ & = \text{Im } M(f) < 0. \end{aligned}$$

Hence, there exists  $L_0 > 1$  such that we have

$$\text{Im} \left[ \frac{1}{2L} \int_{-L}^L f(x) \phi_{R,L}^{p'}(0, x) dx \right] < 0 \quad (3.5)$$

for all  $L \geq L_0$ .

Let  $T > R^2$ . Then, from the weak formulation (3.1) with (3.5), (3.4), and Hölder's inequality, we have

$$\begin{aligned} I_{R,L} &= \text{Im} \left[ \frac{1}{2L} \int_{-L}^L f(x) \phi_{R,L}^{p'}(0, x) dx \right] + \text{Re} \left[ \frac{1}{2L} \int_0^{R^2} \int_{-L}^L u \left( -i \partial_t (\phi_{R,L}^{p'}) + \partial_x^2 (\phi_{R,L}^{p'}) \right) dx dt \right] \\ &< \frac{1}{2L} \int_0^{R^2} \int_{-L}^L |u \cdot \partial_t (\phi_{R,L}^{p'})| dx dt + \frac{1}{2L} \int_0^{R^2} \int_{-L}^L |u \cdot \partial_x^2 (\phi_{R,L}^{p'})| dx dt \\ &\lesssim \frac{1}{R^2} \cdot \frac{1}{2L} \int_{\frac{R^2}{2}}^{R^2} \int_{-L}^L |u(t, x)| \eta_L(x) \phi_{R,L}^{p'-1}(t, x) \left| \partial_t \theta\left(\frac{t}{R^2}\right) \right| dx dt \\ &\quad + \frac{1}{2L} \int_{\frac{R^2}{2}}^{R^2} \int_{-L}^L |u(t, x)| \theta_R(t) \phi_L^{p'-1}(t, x) \left( \frac{|\partial_x \eta_L(x)|^2}{\eta_L(x)} + |\partial_x^2 \eta_L(x)| \right) dx dt \\ &\lesssim \left( \frac{1}{R^2} + \frac{1}{L^{2-2\varepsilon}} \right) \left( \frac{1}{2L} \int_{\frac{R^2}{2}}^{R^2} \int_{-L}^L 1 dx dt \right)^{\frac{1}{p'}} \left( \frac{1}{2L} \int_{\frac{R^2}{2}}^{R^2} \int_{-L}^L |u(x, t)|^p \phi_{R,L}^{p'}(x, t) dx dt \right)^{\frac{1}{p}} \\ &\lesssim (R^{-\frac{2}{p}} + R^{\frac{2}{p'}} L^{-2+2\varepsilon}) I_{R,L}^{\frac{1}{p}} \quad (3.6) \end{aligned}$$

for all  $L \geq L_0$ . Hence, noting that  $p > 1$  and  $\varepsilon \in (0, 1)$ , we have

$$\mathbf{I}_{R,L} \lesssim R^{-\frac{2}{p-1}} + R^2 L^{-\frac{2p}{p-1}(1-\varepsilon)} \leq C < \infty,$$

as long as

$$L \gg \max \left( R^{\frac{p-1}{(1-\varepsilon)p}}, L_0 \right) \quad \text{and} \quad R > 1. \quad (3.7)$$

Since  $\phi_{R,L}(t, x) \equiv 1$  on  $[0, \frac{R^2}{2}) \times [-\frac{L}{2}, \frac{L}{2}]$  for  $L \gg 1$ , we have

$$\int_0^{\frac{R^2}{2}} \frac{1}{2L} \int_{-\frac{L}{2}}^{\frac{L}{2}} |u|^p dx dt \lesssim R^{-\frac{2}{p-1}} + R^2 L^{-\frac{2p}{p-1}(1-\varepsilon)} \leq C < \infty, \quad (3.8)$$

independent of  $L, R \gg 1$ , satisfying (3.7). Since  $u \in L^\infty([0, T]; \mathcal{A}_\omega(\mathbb{R}))$ , we have

$$\frac{1}{2L} \int_{-\frac{L}{2}}^{\frac{L}{2}} |u(t)|^p dx \leq \frac{1}{2} \|u\|_{L^\infty([0, T]; \mathcal{A}_\omega(\mathbb{R}))}^p$$

for all  $L > 0$  and  $t \in [0, T)$ . Then, by Dominated Convergence Theorem, we have

$$\lim_{L \rightarrow \infty} \int_0^{\frac{R^2}{2}} \frac{1}{2L} \int_{-\frac{L}{2}}^{\frac{L}{2}} |u(t)|^p dx dt = \int_0^{\frac{R^2}{2}} \lim_{L \rightarrow \infty} \frac{1}{2L} \int_{-\frac{L}{2}}^{\frac{L}{2}} |u(t)|^p dx dt = \frac{1}{2} \int_0^{\frac{R^2}{2}} M(|u(t)|^p) dt$$

for every fixed  $R > 1$ . Hence, by Monotone Convergence Theorem with (3.8), we obtain

$$\begin{aligned} \int_0^\infty M(|u(t)|^p) dt &= \lim_{R \rightarrow \infty} \int_0^{\frac{R^2}{2}} M(|u(t)|^p) dt \\ &= \lim_{R \rightarrow \infty} \lim_{L \rightarrow \infty} \int_0^{\frac{R^2}{2}} \frac{1}{2L} \int_{-\frac{L}{2}}^{\frac{L}{2}} |u(t)|^p dx dt = 0. \end{aligned}$$

Therefore, by Lemma 1.3, we conclude that  $u(t) = 0$  for almost every  $t$ .  $\square$

**Remark 3.4.** Suppose  $\operatorname{Re} \lambda > 0$  and  $\operatorname{Im} M(f) > 0$ . Let  $u$  be a global weak solution to (1.15) on  $[0, \infty)$  with  $u|_{t=0} = f$ . Then, by repeating the computation in (3.6) and taking the limits of both sides as  $L \rightarrow \infty$  with Dominated Convergence Theorem as above, we

$$\mathbf{I}_R \leq \operatorname{Im} M(f) + C R^{-\frac{2}{p}} \mathbf{I}_R^{\frac{1}{p}}.$$

where

$$\mathbf{I}_R := \operatorname{Re} \lambda \int_0^{R^2} M(|u(t)|^p) \theta_R^{p'}(t) dt.$$

Then, by the continuity argument and Fatou's lemma, we obtain

$$\int_0^\infty M(|u(t)|^p) dt < \infty. \quad (3.9)$$

Hence, it follows from Lemma 1.3 that any global solution on  $[0, \infty)$  must go to 0 as  $t \rightarrow \infty$  in some averaged sense. In view of Proposition 3.3, the same conclusion holds for a global solution  $u \in C([0, \infty); \mathcal{A}_\omega(\mathbb{R}))$  satisfying the Duhamel formulation (3.2).

Finally, we present the proof of Proposition 3.3.



*Proof of Proposition 3.3.* Let  $u$  be a solution in  $C([0, T]; \mathcal{A}_\omega(\mathbb{R}))$ , satisfying the Duhamel formulation (3.2). Write  $u(t) = S(t)f + \mathcal{D}(u)(t)$ , where  $\mathcal{D}(u)$  is given by

$$\mathcal{D}(u)(t) = -i \int_0^t S(t-t') \lambda |u(t')|^p dt'.$$

In the following,  $\phi$  denotes a test function in  $C_c^\infty((-\infty, T) \times \mathbb{R})$ . We use  $W_T^{s,p}$  and  $L_T^p$  to denote  $W_t^{s,p}([0, T])$  and  $L_t^p([0, T])$ , respectively.

First, we show that the linear part  $S(t)f$  satisfies

$$\int_0^T \int_{\mathbb{R}} S(t)f \left( -i\partial_t \phi + \partial_x^2 \phi \right) dx dt = i \int_{\mathbb{R}} f(x) \phi(0, x) dx. \quad (3.10)$$

Given  $f \in \mathcal{A}_\omega(\mathbb{R})$ , define  $f_N$  by (2.2). Then, the corresponding linear solution  $S(t)f_N$  is given by (2.8). By integration by parts, we have

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} S(t)f_N \left( -i\partial_t \phi + \partial_x^2 \phi \right) dx dt \\ = \sum_{\mathbf{n} \in B_N} \widehat{f}(\omega \cdot \mathbf{n}) \int_0^T \int_{\mathbb{R}} e^{-i(\omega \cdot \mathbf{n})^2 t} e^{i(\omega \cdot \mathbf{n})x} \left( -i\partial_t \phi + \partial_x^2 \phi \right) dx dt \\ = i \sum_{\mathbf{n} \in B_N} \widehat{f}(\omega \cdot \mathbf{n}) \int_{\mathbb{R}} e^{i(\omega \cdot \mathbf{n})x} \phi(0, x) dx = i \int_{\mathbb{R}} f_N(x) \phi(0, x) dx. \end{aligned} \quad (3.11)$$

Recall that  $f_N$  converges to  $f$  in  $\mathcal{A}_\omega(\mathbb{R})$  and in  $L^\infty(\mathbb{R})$ . Then, by Hölder's inequality and Lemma 2.2 (ii), we have

$$\begin{aligned} \left| \int_0^T \int_{\mathbb{R}} (S(t)f - S(t)f_N) \left( -i\partial_t \phi + \partial_x^2 \phi \right) dx dt \right| \\ \leq \|S(t)(f - f_N)\|_{L_{t,x}^\infty} (\|\phi\|_{W_T^{1,1}L_x^1} + \|\phi\|_{L_T^1W_x^{2,1}}) \\ \leq \|f - f_N\|_{\mathcal{A}_\omega(\mathbb{R})} (\|\phi\|_{W_T^{1,1}L_x^1} + \|\phi\|_{L_T^1W_x^{2,1}}) \longrightarrow 0, \end{aligned} \quad (3.12)$$

as  $N \rightarrow \infty$ . Similarly, we have

$$\left| \int_{\mathbb{R}} (f(x) - f_N(x)) \phi(0, x) dx \right| \leq \|f - f_N\|_{L^\infty} \|\phi(0, x)\|_{L_x^1} \longrightarrow 0, \quad (3.13)$$

as  $N \rightarrow \infty$ . Hence, (3.10) follows from (3.11), (3.12), and (3.13).

Since  $u \in C([0, T]; \mathcal{A}_\omega(\mathbb{R}))$ , we have  $\lambda |u|^p \in C([0, T]; \mathcal{A}_\omega(\mathbb{R}))$ . Given an enumeration  $\{r_j\}_{j=1}^\infty$  of  $\mathbb{Z}^\mathbb{N}$ , define  $\mathcal{D}_N$  by

$$\mathcal{D}_N(t, x) = -i\lambda \sum_{\mathbf{n} \in B_N} \int_0^t \widehat{|u|^p}(t', \omega \cdot \mathbf{n}) e^{-i(\omega \cdot \mathbf{n})^2(t-t')} dt' e^{i(\omega \cdot \mathbf{n})x}$$

for  $t \in [0, T]$ , where  $B_N = \{r_j\}_{j=1}^N$ ,  $N \in \mathbb{N}$ . Since  $\mathcal{D}_N$  is a finite linear combination of smooth functions, noting that  $\mathcal{D}_N(0, x) = \phi(T, x) = \partial_x \phi(T, x) = 0$  for all  $x \in \mathbb{R}$ , integration by parts yields

$$\int_0^T \int_{\mathbb{R}} \mathcal{D}_N \left( -i\partial_t \phi + \partial_x^2 \phi \right) dx dt = \lambda \int_0^T \int_{\mathbb{R}} U_N \phi dx dt,$$

where  $U_N$  is defined by

$$U_N(t, x) = \sum_{\mathbf{n} \in B_N} \widehat{|u|^p}(t, \boldsymbol{\omega} \cdot \mathbf{n}) e^{i(\boldsymbol{\omega} \cdot \mathbf{n})x}.$$

Then, by proceeding as in the proof of Lemma 2.3 (i), we obtain

$$\int_0^T \int_{\mathbb{R}} \mathcal{D}(u) \left( -i\partial_t \phi + \partial_x^2 \phi \right) dx dt = \lambda \int_0^T \int_{\mathbb{R}} |u|^p \phi dx dt. \quad (3.14)$$

The identity (3.1) follows from (3.10) and (3.14).  $\square$

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